Classical Geometric Interaction- picture-like Description [*]

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Abstract

In order to get the classical analogue of quantum interaction picture in classical symplectic geometric description, the space of solutions of free equations of motion is suggested to replace the phase space in T^*Q description or the space of motions in usual classical symplectic geometric description. The way to determine measured values of observables in this scheme is worked out.

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It is well known that there exist many different pictures in quantum formalism. The most famous and useful ones are Schrödinger, Heisenberg and interaction pictures. During recent

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years a relatively new quantization formulation called geometric quantization has been paid attentions and been considered as so far mathematically most thorough approach to quantization. In short, geometric quantization [1][2] is essentially a globalization of canonical quantization. It is heavily based on the symplectic geometrical description of classical system. In books on the geometric quantization, usually, ones start from a phase space (for example, tangent bundle TQ or cotangent bundle T^*Q of configuration space Q). But, as pointed out in ref.[4], this description is not obviously relativistically covariant in relativistic theory since at the beginning a special time should be chosen. Thus, another description, which uses the space of solutions of the equation of motion (called space of motion M) as the state space[3], has been used to establish covariant geometric formalisms of various relativistic field theories by Crnkovic and Witten et al. [4][5][6][8]. In fact, as pointed in P.21 of the ref.[1], above two descriptions are classical analogues of quantum Schrödinger and Heisenberg pictures respectively. Thus a natural question arises. How to establish a classical analogue of the interaction picture in classical symplectic geometric description since the interaction picture is very important in quantum perturbative calculation. In this paper, we investigate this problem and work out it based on the space of solutions of free equations of motion. We illustrate its relationships with above two classical geometric descriptions are similar to the relationships between the interaction, Schrödinger and Heisenberg Pictures of quantum mechanics. Thus in geometric descriptions the classical and quantum structures in this respect are given in parallel .

Let us consider a conservative dynamical system for which the set of kinematically possible states can be represented by a velocity phase space or tangent bundle TQ. Its dynamical behavior is determined by a Lagrangian $L(q^a, \dot{q}^a)$, which is regular in the following sense, i.e.,

$$\det(\frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}) \neq 0 \tag{1}$$

everywhere. The Hamiltonian h of the system can be defined by

$$h(q^a, \dot{q}^a) = \dot{q}^a \frac{\partial L}{\partial \dot{q}^a} - L. \tag{2}$$

We also require that the Hamiltonian vector field X_h is complete. The equations of motion are

given by the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = 0. \tag{3}$$

In order to transfer to the momentum phase space(or cotangent bundle T^*Q) ones take the Lengendre transformation

$$\rho : TQ \to TQ^* : (q^a, \dot{q}^a) \mapsto (q^a, p_a), \tag{4}$$

where locally

$$p_a = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^a}. \tag{5}$$

In this scheme Hamiltonian becomes $H(q, p) = h(q, \dot{q}(q, p))$. The equations of motion are given by the Hamiltonian equations,

$$\dot{q}^a(t) = \frac{\partial H(q^a(t), p_a(t))}{\partial p_a(t)}, \tag{6}$$

$$\dot{p}_a(t) = -\frac{\partial H(q^a(t), p_a(t))}{\partial q^a(t)}.$$

In order to discuss interaction- picture-like description let us write the Lagrangian into two parts,

$$L = L_0 + L_1, \tag{7}$$

where L_0 is a free Lagrangian[1] and L_1 is the interaction part. Thus the equation of free motion of this system is given by

$$\frac{d}{dt}\left(\frac{\partial L_0(q_0^a(t), \dot{q}_0^a(t))}{\partial \dot{q_0}^a(t)}\right) - \frac{\partial L_0(q_0^a(t), \dot{q}_0^a(t))}{\partial q_0^a(t)} = 0. \tag{8}$$

The manifold of solutions of the equation (8) of free motions M_F can be defined as follows. From now on, q_0 will be used to denote a free solution mapping, which is defined to be

$$q_0 : t \mapsto (q_0^a(t)), \tag{9}$$

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[1]In fact following approach is suitable for any regular L_0. In practice ones often take a Lagrangian whose equation of motion has rigorous solutions as L_0.

where $(q_0^a(t))$ satisfies the equation (8). M_F consists of all solution mappings of the eq.(8). It can be given a topology and made into a manifold by using the values of $q_0^a(t_0)$ and $\dot{q}_0^a(t_0)$ at some particular time t_0 as coordinates. The mapping

$$
\begin{aligned}
\tau_t \quad &: \quad M_F \to TQ \\
&: \quad q_0 \mapsto (q_0^a(t), \dot{q}_0^a(t))
\end{aligned}
\tag{10}
$$

is a diffeomorphism since the free Lagrangian L_0 is regular and X_{h_0} is complete. Note that if $q_0 : t \mapsto (q_0^a(t))$ is a solution of (8), so is $q_0^{a'} : t \mapsto (q_0^{a'}(t) \equiv q_0^a(t+k))$ where k is a constant. Generally speaking, it is a distinct solution so that regarded as a distinct point of M_F, even though the two orbits occupy the same point set in Q.

A tangent vector U to M_F at a solution q_0 is represented by a solution u_0 of the linearized equations of free motion (8)

$$
u_0 : t \longmapsto u_0(t)
$$

and $u_0(t)$ satisfies

$$
\left\{\frac{d}{dt}\left(\frac{\partial^2 L_0}{\partial \dot{q}_0{}^a \partial \dot{q}_0^b} \dot{u}_0^b + \frac{\partial^2 L_0}{\partial \dot{q}_0{}^a \partial q^b} u_0^b\right) - \frac{\partial^2 L_0}{\partial q_0^a \partial \dot{q}_0{}^b} \dot{u}_0{}^b - \frac{\partial^2 L_0}{\partial q_0^a \partial q_0^b} u_0^b\right\}\Biggr|_{\substack{u_0^b = u_0^b(t) \\ q_0^a = q_0^a(t)}} = 0.
\tag{11}
$$

The free part of action is

$$
I_0(t_2, t_1) = \int_{t_1}^{t_2} dt \cdot L_0(q_0^a(t), \dot{q}_0^a(t)),
\tag{12}
$$

where t_1 and t_2 are fixed values of the time. $I_0(t_2, t_1)$ can be considered as a function on M_F when, as shown in (12), the arguments of L_0 are restricted to solutions of the free equation. Thus the derivative of $I_0(t_2, t_1)$ along U is given by

$$
\begin{aligned}
U \circ dI_0 \quad = \quad & \int_{t_1}^{t_2} dt \cdot \left[\frac{\partial L_0(q_0^a(t), \dot{q}_0^a(t))}{\partial q_0^a(t)} u_0^a(t) + \frac{\partial L_0(q_0^a(t), \dot{q}_0^a(t))}{\partial \dot{q}_0{}^a(t)} \dot{u}_0{}^a(t)\right] \\
= \quad & \left[\frac{\partial L_0}{\partial \dot{q}_0{}^a(t)} u_0^a(t)\right]\Biggr|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\, u_0^a(t)\left[\frac{\partial L_0(q_0^a(t), \dot{q}_0^a(t))}{\partial q_0^a(t)} - \frac{d}{dt}\left(\frac{\partial L_0(q_0^a(t), \dot{q}_0^a(t))}{\partial \dot{q}_0{}^a(t)}\right)\right] \tag{13}
\end{aligned}
$$

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Now the final integral in (13) vanishes because q_0 is a solution of the equation (8). Thus we find we can define for each t a 1-form θ_t on M_F by

$$U \circ \theta_t = u_0^a(t) \frac{\partial L_0(q_0^a(t), \dot{q}_0^a(t))}{\partial \dot{q}_0^a(t)}, \tag{14}$$

so that the equation (13) implies that

$$dI_0(t_2, t_1) = \theta_{t_2} - \theta_{t_1}. \tag{15}$$

Since $dI_0(t_2, t_1)$ is an exact form on M_F, we get from (15) that

$$d\theta_{t_2} - d\theta_{t_1} = 0. \tag{16}$$

So the closed form in terms of coordinates $(q_0^a(t), \dot{q}_0^a(t))$

$$\omega_T \equiv d\theta_t = d\frac{\partial L_0(q_0^a(t), \dot{q}_0^a(t))}{\partial \dot{q}_0^a(t)} \wedge dq_0^a(t) \tag{17}$$

on TQ does not depend on t and, its pull-back $\tau_t^* \omega_T$ provides a natural symplectic structure on M_F. From this symplectic structure we can define Hamiltonian vector fields, Poisson bracket etc. on M_F.

Above presentation uses tangent bundle TQ. In practical application the momentum phase space (or cotangent bundle TQ^*) is also often used. Similarly to τ_t in (10), we can define another mapping Π_t

$$\Pi_t \quad : \quad M_F \mapsto T^*Q$$

$$: \quad q_0 \mapsto (q_0^a(t), p_{a0}(t)), \tag{18}$$

i.e.

$$(\hat{q}^a \circ \Pi_t q_0, \hat{P}_a \circ \Pi_t q_0) \quad = \quad (q_0^a(t), p_{a0}(t))$$

$$\equiv \quad \Pi_t q_0. \tag{19}$$

In (18) q_0 is the same solution of free motion in (10). Equivalently, instead of eq.(8), we have $(q_0^a(t), p_0^a(t))$ satisfies the Hamiltonian equations of free motion

$$\dot{q}_0^a(t) \quad = \quad X_{H_0(q_0^a(t), p_{a0}(t))} \, q_0^a(t),$$

$$\dot{p}_{a0}(t) \quad = \quad X_{H_0(q_0^a(t), p_{a0}(t))} \, p_{a0}(t). \tag{20}$$

If we take $(q_0^a(t_0), p_{a0}(t_0))$ at a given time t_0 as coordinates of M_F, by using them we can write the symplectic 2-form as

$$\omega_c = dp_{a0}(t_0) \wedge dq_0^a(t_0). \tag{21}$$

It is easy to prove that ω_c does not depend on t_0 since the transformation $(q_0^a(t_0), p_{0a}(t_0)) \rightarrow (q_0^a(t), p_{0a}(t))$ is a canonical transformation. Therefore its pullback $\Pi_{t_o}^* \omega_c$ is well defined on M_F and provides a symplectic 2- forms. Comparing eqs.(21) and (17), we find $\Pi_{t_o}^* \omega_c = \tau_t^* \omega$.

Now let us turn to discuss how to determine the classical measured values of observables at time t in this scheme. First of all we like to point out that $H_1(q_0^a(t), p_{a0}(t)) \in R$ depends on not only solution q_0, but also the time t. So if we define a function $(\tau_t^F H_1)$ on M_F as follows

$$(\Pi_t^* H_1) \quad : \quad M_F \rightarrow R$$
$$\quad : \quad q_0 \mapsto (\Pi_t^* H_1) q_0 \equiv H_1(q_0(t), p_0(t)). \tag{22}$$

it is $t-$dependent. From eqs. (20) and (22), the $t-$ dependence of the quantity $(\Pi_t^* H_1)$ is determined by the free Hamiltonian. Then from its Hamiltonian vector field $X_{\Pi_t^* H_1}$ ones can generate a curve on M_F

$$\frac{d}{dt} q_0^{(t)} = X_{\Pi_t^* H_1} q_0^{(t)} \tag{23}$$

with t as the parameter of the curve. Here $q_0^{(t)}$ represents the state in such description. Note since $\Pi_t^* H_1$ is dependent on parameter t , so does the generator $X_{\Pi_t^* H_1}$. Furthermore from above settings we define

$$(q^a(t), p_a(t)) \equiv (q_0^{a(t)}(t), p_{a0}^{(t)}(t)) = \Pi_t q_0^{(t)} = \Pi_t q_0^{(s)}\Big|_{s=t}. \tag{24}$$

We shall prove that $(q^a(t), p_a(t))$ defined here satisfies the Hamiltonian equations (6). In fact,

$$\begin{aligned}
(\frac{dq^a(t)}{dt}, \frac{dp_a(t)}{dt}) &= \frac{d}{dt}(\Pi_t q_0^{(t)}) = \frac{d}{dt}(q^{a(t)}(t), p_a^{(t)}(t)) \\
&= (\frac{dq_0^{a(s)}(t)}{dt}, \frac{dp_{a0}^{(s)}(t)}{dt})\Big|_{s=t} + (\frac{dq_0^{a(s)}(t)}{ds}, \frac{dp_{a0}^{(s)}(t)}{ds})\Big|_{s=t} \\
&= X_{H_0(q_{0(t)}^{a(s)}, p_{a0}^{(s)}(t))} \Pi_t q_0^{(s)}\Big|_{s=t} + \Pi_t X_{\Pi_t^* H_1(q_0^{(s)})} q_0^{(s)}\Big|_{s=t} \\
&= X_{H_0(q_0^{a(s)}(t), p_{a0}^{(s)}(t))} q_0^{(s)}(t)\Big|_{s=t} + \Pi_t X_{\Pi_t^* H_1(q_0^{(s)})} \Pi_t^{-1} \Pi_t q_0^{(s)}\Big|_{s=t}
\end{aligned}$$

6

$$
\begin{aligned}
&= (X_{H_0(q_0^{a(t)}(t),p_{a0}^{(t)}(t))}q_0^{a(t)}(t), X_{H_0(q_0^{a(t)}(t),p_{a0}^{(t)}(t))}p_{a0}^{(t)}(t)) \\
&\quad + (X_{H_1(q_0^{a(t)}(t),p_{a0}^{(t)}(t))}q_0^{a(t)}(t), X_{H_1(q_0^{a(t)}(t),p_{a0}^{(t)}(t))}p_{a0}^{(t)}(t)) \\
&= (X_{H(q_0^{a(t)}(t),p_{a0}^{(t)}(t))}q_0^{a(t)}(t), X_{H(q_0^{a(t)}(t),p_{a0}^{(t)}(t))}p_{a0}^{(t)}(t)) \\
&= (X_{H(q^a(t),p_a(t))}q^a(t), X_{H(q^a(t),p_a(t))}p_a(t)),
\end{aligned}
\tag{25}
$$

where in the fifth step, we have used the relations $\Pi_t q_0^{(s)} = q_0^{(s)}(t)$ and $\Pi_t X_{\Pi_t^* H_1(q_0^{(s)})}\Pi_t^{-1} = X_{H_1(q_0^{a(s)}(t),p_{a0}^{(s)}(t))}$. The final line in (25) is nothing but the Hamiltonian equations(6). Therefore the constructions(24) truly describes the whole evolution of measured values of q and p.

Now let us compare above M_F description with phase space T^*Q and motion space M descriptions. As explained in $P.21$ of the ref.[1], there exit changes in point of view. In the phase space T^*Q, the system evolves by moving along an integral curve of X_H. In the space of motions M, the state of system is represented by a fixed point of M, while the physical observables are represented by time-dependent functions $\hat{q}{}^a \circ \tau_t, \hat{p}_a \circ \tau_t, ...$ on M where $\tau_t : M \to T^*Q : q \mapsto (q^a(t), p_a(t))$. In the space of free motions M_F, the state of system evolves by moving along a curve generated by interaction Hamiltonian, while observables also are time-dependent functions $\hat{q}{}^a \circ \Pi_t, \hat{p}_a \circ \Pi_t, ...$ on M_F. Their actual classical measured values at t is determined by the combinational effects. These distinctions are just analogues of distinctions between Schrödinger, Heisenberg and the interaction pictures in quantum mechanics.

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